polynomial_bilevel

Let $x \in \mathbb{R}^{n_x}, y \in \mathbb{R}^{n_y}$ be the lower and upper level decision variables, respectively. Given a data parametrisation $m \in \mathbb{N}$, the number of terms; $c \in \mathbb{R}^m$ a vector of coefficients; $p \in \mathbb{Z}^{n_x,m}$ a matrix of exponents for x; and $p \in \mathbb{Z}^{m,n_y}$ a matrix of exponents for y then we define the polynomial as:

$$\mathcal{P}_{c,p,q}(x,y) \coloneqq \sum_{K=1}^{m} c_k \prod_{i=1}^{n_x} x_i^{p_{k,i}} \prod_{j=1}^{n_y} y^{q_{k,j}}.$$

We make no restriction on the degree or number of terms in these polynomials. Given a sequence of polynomials $\mathcal{P}^0_{\text{up}}, \mathcal{P}^k_{\text{up}}, \mathcal{P}^0_{\text{lo}}, \mathcal{P}^k_{\text{lo}}$, the The *Polynomial Bilevel Program* (PBP) is defined as:

$$\begin{aligned}
\underset{x\in\mathbb{R}^{n_x}, y\in\mathbb{R}^{n_y}}{\text{minimise}} \quad & \mathcal{P}^0_{\text{up}}(x,y) && \text{(PBP)} \\
\text{subject to} \quad & \mathcal{P}^k_{\text{up}}(x,y) \geq 0 \qquad \text{for } k = 1,2,\dots, \\
& A_{\text{up}}x + B_{\text{up}}y \geq r_{\text{up}}, \\
& \underline{x} \leq x \leq \bar{x}, \\
& y \in \underset{y\in\mathbb{R}^{n_y}}{\arg\min} \left\{ \begin{array}{lll} \mathcal{P}^0_{\text{lo}}(x,y) & & \\ \text{s.t.} & \mathcal{P}^k_{\text{lo}}(x,y) \geq 0, & \text{for } k = 1,2,\dots, \\ & A_{\text{lo}}x + B_{\text{lo}}y \geq r_{\text{lo}}, & \\ & \underline{y} \leq y \leq \bar{y}. & \end{array} \right.
\end{aligned}$$

The (PBP) is represented neatly by a JSON file. Below, we demonstrate how the lower-level objective function of Mitsos and Barton's test example [1, Example 3.10] is represented in JSON by the parameters (m, c, p, q) of its polynomial.

$$f(x,y) = \quad 16xy^4 \quad +2xy^3 \quad -8xy^2 \quad -1.5xy \quad +0.5x.$$

```
"f": {
    "m": 5,
    "c":   [  16.0,  2.0,  -8.0,  -1.5,  0.5  ],
    "p":  [[  1,     1,    1,     1,     1    ]],
    "q":  [[  4,     3,    2,     1,     0    ]]
}
```

This allows for easy derivative calculations using the formal derivative. Furthermore, the work of Nie et al. [2, 3] suggests a bespoke numerical method for solving the polynomial bilevel program.

References

[1] Alexander Mitsos and Paul I Barton. A test set for bilevel programs, 2006.

[2] Jiawang Nie, Li Wang, and Jane J. Ye. Bilevel polynomial programs and semidefinite relaxation methods. *SIAM Journal on Optimization*, 27(3):1728–1757, 2017.

[3] Jiawang Nie, Li Wang, Jane J. Ye, and Suhan Zhong. A lagrange multiplier expression method for bilevel polynomial optimization. *SIAM Journal on Optimization*, 31(3):2368–2395, 2021.